Indian Statistical Institute, Bangalore Centre
B.Math. (III Year) : 2015-2016
Semester II : Mid-Semestral Examination
Probability III (Stochastic Processes)

26.02.2016 Time: $2\frac{1}{2}$ hours. Maximum Marks : 80

1. (15 marks) Let $\{X_n\}$ be the simple random walk on $\mathbb{Z}$ with $P_{i,i+1} = p,\ P_{i,i-1} = 1-p,\ i \in \mathbb{Z}$, where $0 < p < 1$. Let $D \geq 1$ be an integer. Find $P(\{X_n\}$ reaches 0 before reaching $D \mid X_0 = i)$ for $0 \leq i \leq D$.

2. (15 + 15 = 30 marks) $\{X_n : n = 0, 1, 2, \cdots\}$ is a Markov chain on $S = \{1, 2, 3, 4, 5\}$ with the transition probability matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

   (i) Find the set of transient states, and the irreducible closed set(s) of recurrent states.

   (ii) Find the probability of eventual absorption in the irreducible closed set(s) of recurrent states.

3. (10 + 10 = 20 marks) (i) Show that a state y is positive recurrent if and only if $\lim_{n\to\infty} \frac{1}{n} G_n(y, y) > 0$.

   (ii) If a state y is null recurrent show that $\lim_{n\to\infty} \frac{1}{n} G_n(x, y) = 0$ for any state x. Is the converse true? (Here $G_n(x, y)$ is the expected number of visits to y during $\{1, 2, \cdots n\}$, starting from x.)

4. (15 marks) A is a $k \times k$ transition probability matrix such that $A^\dagger$ is also a transition probability matrix. Find a stationary probability distribution for A. Is it unique? Justify.